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The four-color theorem in the service of Euclidean distance Into the $(n_0, \rho_0) - R^2$ graphs

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ABSTRACT: The four-color theorem in graph theory, [1,3 and 4] states that every planer graph is 4 - coloring (i.e., all the vertices of the graph can be colored in 4 colors in which no two vertices share the same edge have the same color).

In this paper, we will be based on this theorem, we will discover a feature in graphs, which are given in the Euclidean plane, and their edge are defined by Euclidean distances (see Theorem 1 and Theorem 2 in this paper).

Keywords: The four-color theorem, chromatic number, planar graph, a clique in graph.

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1. INTRODUCTION

Definition 1: The Euclidean distance between any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ on the plane is marked with $d_2(x, y)$, and is defined to be $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

Definition 2: A connected graph is a graph in which for each pair of vertices x and y there exists a path (or walk) beginning at x and ending at y. When a path of length s is a sequence of vertices $x_1, x_2, x_3, \dots, x_s$, in which for all $i > 1$, x_i is adjacent to x_{i+1}.

Definition 3: A planar graph is a graph that we can draw on the plane without cutting between its edges.

Definition 4: A graph $G: (V, E)$ is called k - coloring if we can color all his vertices k colors so that no two vertices share the same edge.

The smallest number of colors needed to color a graph G is called its chromatic number, and often denoted by χ (G).

Definition 5: In a graph $G: (V, E)$ a subset C of V is called a clique in G, if every two distinct vertices are adjacent. i.e. every clique in G is a sub-graph which is isometric to k_m, when K_m; the complete graph, with m vertexes such that every pair of vertices is connected by an edge.

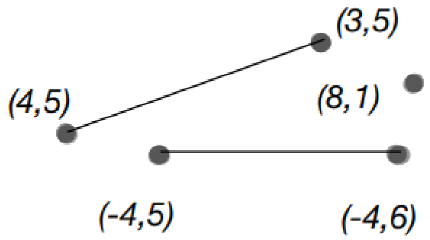
A maximum clique of a graph G is a clique, in which there is no clique with more vertices. Moreover, the clique number $\omega(G)$, of a graph is the number of vertices in a maximum clique in G .

An independent set (or co-clique set) or (anti-clique set), is a subset of vertices S of V in $G: (V, E)$, that no two of its vertices are adjacent. (i.e. every anti-clique in G is a sub-graph N_m of G when N_m denotes the empty graph, n vertexes with no edges).

To facilitate the explanation and proof of the new result in this paper, I will define for $n_0 \epsilon N$, $n_0 \geq 2$, and for a given positive number $\rho_0 \geq 0$, family of graphs, which we gave it the name $(n_0, \rho_0) - R^2$ graphs, and denote it with $G_{\rho_0}^{n_0}(V_0, E_0)$, as follows:
$V_0 = \{(x_i)_{i=1}^{n_0} : x_i \in R^2, d_2(x_i, x_j) \geq \rho_0, 1 \leq i, j \leq n_0\}$, this mean that the vertex group in each $G_{\rho_0}^{n_0}$ is a collection of n points in the plane, in which The Euclidean distance between any two points is at least ρ_0.

$E_0 = \{(x_i, x_j): d_2(x_i, x_j) = \rho_0, 1 \le i, j \le n\, n_0\}$, this means that there is an adage between any two vertices, which their Euclidean distance exactly ρ_0.
It is clear that $G_{\rho_0}^{n_0}$ who get the maximum number of edges is exactly K_{n_0}, the complete graph, and $G_{\rho_0}^{n_0}$ who get the minimum number of edges is exactly N_{n_0} the empty graph.

Example 1:
A graph of the family G_1^5 described in figure 1, an .



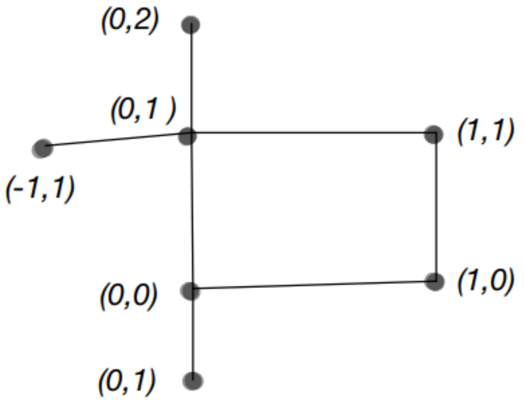
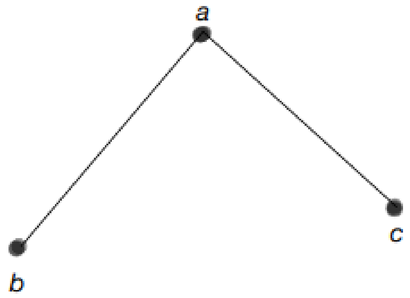


Figure 1

Figure 2

Example 2:
The set $V = \{(0,0), (0,1), (0,2), (4,10)\}$, cannot describe any graph in G_1^4.

Example 3:
The set $V = \{a: (0,0), b: (-1, \sqrt{2}), c: (1, \sqrt{2})\}$, describes a graph in G_3^3.



2. NEW RESULT

Our main result is presented in the following theorems:

Theorem 1:
For every positive number $\rho_0 \ge 0$ and for every $n_0 \in N, n_0 \ge 2$, all $(n_0, \rho_0) - R^2$ graph, (i.e. the family $G_{\rho_0}^{n_0}(V_0, E_0)$) is a planar family.

Theorem 2:
For every $\rho_j \ge 0$, and for every $n_j \in N, n_j \ge 2$, in each $(n_J, \rho_J) - R^2$ graph, i.e., in each $G(V,E) \in G_{\rho_j}^{n_j}(V_j, E_j)$, there is a subset $\tilde{V}, \tilde{V} \subset V$ which contains at least $\frac{n_j}{4}$ vertices, so that the Euclidean distance between any two vertices x and y in $\tilde{V}$ is $d_2(x, y) > \rho_j$.

Proof of the Theorem 1:
Suppose in the negative way, that there is a graph G from the family $G_{\rho_0}^{n_0}(V_0, E_0)$ is not planar, for a given positive number $\rho_0 \ge 0$, and for $n_0 \in N, n_0 \ge 2$, therefore, we can assume, that there are two edges in

the graph G that cut each other, we symbolize them $u = (x_1, y_1)$ and $v = (x_2, y_2)$. (Let us notice that all of the points $x_1, y_1, x_2, y_2 \in R^2$).
Since that $u, v \in V_0$, then $d_2(x_1, y_1) = \rho_0$, and $d_2(x_2, y_2) = \rho_0$.
We will mark the point of intersection between u and v in z.
Clearly, we can assume, without losing generality, that x_1 is closer to z than y_1, and that z is closer to x_2 than y_2, i.e., $d_2(z, y_1) > d_2(z, x_1)$ and $d_2(z, y_2) > d_2(z, x_2)$. (See figure 3).

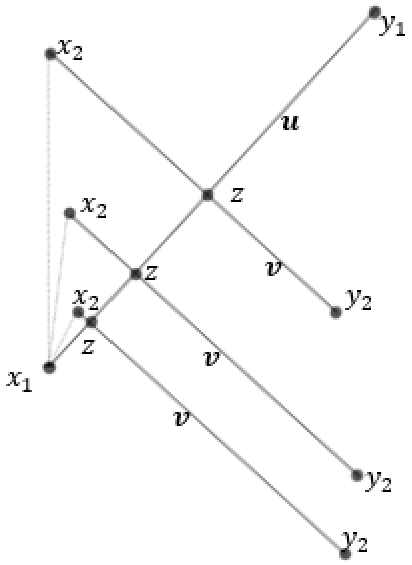


Figure 3

In any possible case for the triangle, $\Delta x_1 z x_2$ it happens that $d_2(z, x_1) < \rho_0$ and $d_2(z, x_2) < \rho_0$, even if the point z is in the middle of u and in the middle of v, and that is exactly contradictory to the fact that there are no two vertices in graph G whose distance is less than ρ_0, hence each
$\rho_0 - R^2$ Graph is planar, i.e., the family $G_{\rho_0}^{n_0}(V_0, E_0)$ is a planar family.

This completes the proof of theorem 1.

Proof of the Theorem 2:
Let $\rho_j \geq 0$, and let $n_j \in N, n_j \geq 2$.
Suppose in the negative way that there is a $(n_J,\ \rho_J) - R^2$ graph, $G(V, E) \in G_{\rho_j}^{n_j}(V_j, E_j)$ which does not have a subset of vertices $\tilde{V}$, $\tilde{V} \subset V$ that contains at least $\frac{n_j}{4}$ vertices, in which that the Euclidean distance between any two vertices x and y in $\tilde{V}$ is $d_2(x, y) > \rho_j$, then the graph $G(V, E)$ does not have an anti-clique set, which contains at least $\frac{n_j}{4}$ vertices.
Divide now the set vertices V into anti-cliques $V_1,\ V_2, \ldots, V_l$ whose subsets in $G(V, E)$.

In general, it is always true that this division is in fact a possibility of coloring of the set V; this is possible because if we gave every anti-clique set V_i the same color, so no two neighboring vertices in V will get the same color. In addition, the minimum number of the anti-clique in such division is exactly the chromatic number χ (G), which is the smallest number of colors needed to color all the vertices so that no two vertices share the same edge.

Since each anti-clique set, in $G(V, E)$, contains at least $\frac{n_j}{4}$ vertices, the minimum number of anti-cliques in the previous division will be bigger than $\frac{n_j}{\frac{n_j}{4}}$, that means $l > \frac{n_j}{\frac{n_j}{4}}$, and also

$\chi\,(G) > \frac{n_j}{\frac{n_j}{4}}$.

Since for every $n_j \geq 2$, it's always true that:

$$\frac{n_j}{\frac{n_j - 1}{4}} \geq \frac{n_j}{\frac{n_j}{4}}$$

Therefore:

$$\chi\,(G) > \frac{n_j}{\frac{n_j - 1}{4}}$$

Then:

$$\chi\,(G) > \frac{4n_j}{n_j - 1} = 4$$

Now, since that $G(V,E) \in G_{\rho_j}^{n_j}(V_j, E_j)$, from theorem 1, we get that $G(V,E)$ is a planar graph, and from the four-color theorem, which states that every planer graph is 4 – coloring, contrary to what we received that $\chi\,(G) > 4$.

This completes the proof of theorem 2

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