

ORDER- GRAPH OVER A GROUP

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ABSTRACT: Graph theory is widely used to solve and understand problems in various fields of mathematics; here we will present a new relationship between a graph and a group (The reference here to the group is the foundation stone structure in modern algebra and algebraic structures).

Keywords: Cyclic group, order group, Chromatic number and planar graph.

By defining a new graph over a group using a feature we can infer new features on both the graph and the algebraic structure if this feature exists.

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I hope there will be a sequel to this topic that will be presented here, as well as to figure out new results.

1. INTRODUCTION

Definition 1: An ordered pair $(G, *)$ where G denotes a set of element and $*$ denotes a [binary operation](#) between them, the order pair will be a group if the [binary operation](#) is closed on G , the trait of associativity exists and the existence identity element (e_G) of the group and every element g in G has his inverse g^{-1} .

Definition 2: Two groups are called isomorphic between them if there is a function 1-1, a surjective function and to keep the operation.

Definition 3: The order of a group, which usually denoted by $|G|$, is defined as the number of its members.

Definition 4: For every $g \in G$, we will define $\langle g \rangle$ to be the set created by g ,
 $\langle g \rangle = \{g^n : g \in G, n \in Z\}$.

Definition 5: A group is called cyclic group if it is formed by a single element.

Definition 6: The order of element $g \in G$, which usually denoted by $O(g)$ is defined as the minimum natural number $n \in N$ in which $g^n = e_G$.

Remark 1: We will mention the following features of the order, [8], for all $g, g_1, g_2 \in G$,

$$\begin{aligned} g^{|G|} &= e_G \\ o(e_G) &= 1 \\ O(g) &= O(g^{-1}) \\ O(g_1 * g_2) &= O(g_2 * g_1) \\ O(g_1) &= O(g_2^{-1} * g_1 * g_2) \\ \text{and } O(g) & \mid |G|. \end{aligned}$$

Remark 2: We denote by $(Z_n, +_n)$, $n \geq 2$ for the cyclic group modulo n , when $Z_n = \{0, 1, 2, \dots, n-1\}$, and the connection modulo n its [binary operation](#). It is known that each cycle's group of order n is isomorphic to $(Z_n, +_n)$. In addition, we know that every $g \in G$, holds: $O(g) = \frac{n}{\gcd(n, g)}$ in every $(Z_n, +_n)$, [3].

Definition 7: A graph $G: (V, E)$ is called k - coloring if we can color all its vertices in k colors so that no two vertices share the same edge.

The smallest number of colors needed to color a graph G is called the chromatic number of G , and often denoted by $\chi(G)$.

Definition 7: A planar graph is a graph that we can draw on the plane without cutting between its edges.

Definition 8: A bondage element of a graph is [a concatenation maximum subgraph](#), which is, its vertex group set, V , have a path between each of its two vertexes, every vertex from V have no path to any other vertex outside V .

New Definition:

Let $(G, *)$ be any group, we will define graph $\Omega_{(G,*)}: (V_\Omega, E_\Omega)$ as follows, $V_\Omega = G$ and two vertices g_1, g_2 of V_Ω are connected on edge if and only if $O(g_1) = O(g_2)$.

We will called to the graph $\Omega_{(G,*)}: (V_\Omega, E_\Omega)$. **The order- graph over the group $(G, *)$,**

Example 1:

Figure 1, illustrates the Order- graph $\Omega_{(Z_{12}, +_{12})}$ over the group $(Z_{12}, +_{12})$.

From that, for every $g \in G$ holds $O(g) = \frac{n}{\gcd(n,g)}$ in every $(Z_n, +_n)$, it is clear we got for every element in $(Z_{12}, +_{12})$ that:

- $O(0) = 1$
- $O(6) = 2$
- $O(1) = O(5) = O(7) = O(11) = 12$
- $O(2) = O(10) = 6$
- $O(3) = O(9) = 4$
- $O(4) = O(8) = 3$

Hence The Order- graph $\Omega_{(Z_{12}, +_{12})}$ over the group $(Z_{12}, +_{12})$, and:

$$V_\Omega = \{0, 1, 2, \dots, 11\} = Z_{12}$$

$$E_\Omega = \{(1,5), (1,7), (1,11), (5,7), (5,11), (7,11), (2,10), (3,9), (4,8)\}.$$

Figure 2, illustrates the Order- graph $\Omega_{(S_3, o)}$ over the symmetric group (S_3, o) , (The group that contains all the permutations from the numbers from 1 to n , and the operation is function composition.

$S_3 = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$. The following table describes the result of the operation between the members of (S_3, o) , [9].

o	(1, 2, 3)	(1, 3, 2)	(2, 1, 3)	(2, 3, 1)	(3, 1, 2)	(3, 2, 1)
(1, 2, 3)	(1,2,3)	(1,3,2)	(2,1,3)	(2,3,1)	(3,1,2)	(3,2,1)
(1, 3, 2)	(1,3,2)	(1,2,3)	(3,1,2)	(3,2,1)	(2,1,3)	(2,3,1)
(2, 1, 3)	(2,1,3)	(2,3,1)	(1,2,3)	(1,3,2)	(3,2,1)	(3,1,2)
(2, 3, 1)	(2,3,1)	(2,1,3)	(3,2,1)	(3,1,2)	(1,2,3)	(1,3,2)
(3, 1, 2)	(3,1,2)	(3,2,1)	(1,3,2)	(1,2,3)	(2,3,1)	(2,1,3)
(3, 2, 1)	(3,2,1)	(3,1,2)	(2,3,1)	(2,1,3)	(1,3,2)	(1,2,3)

It is clear we got for every element in (S_3, o) that:

- $O((1,2,3)) = 1$
- $O((2,3,1), (3,1,2)) = 2$
- $O((3,2,1), (1,3,2), (2,1,3)) = 3.$

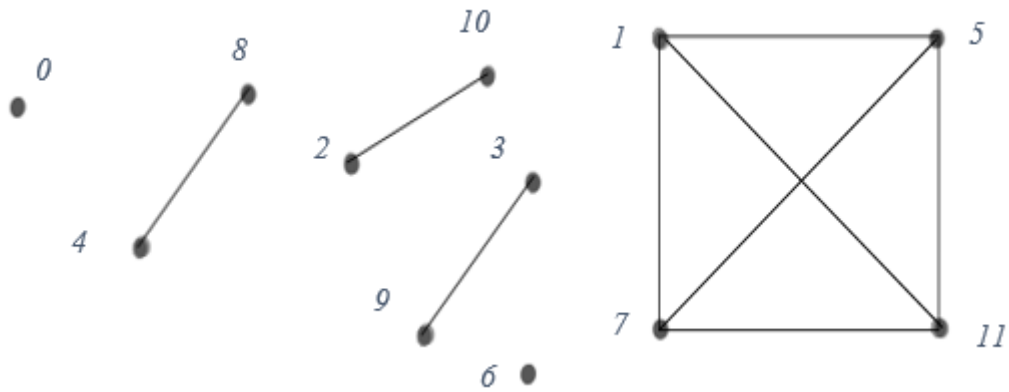


Figure 1

The order- graph $\Omega_{(Z_{12}, +_{12})}$ over the group $(Z_{12}, +_{12})$.

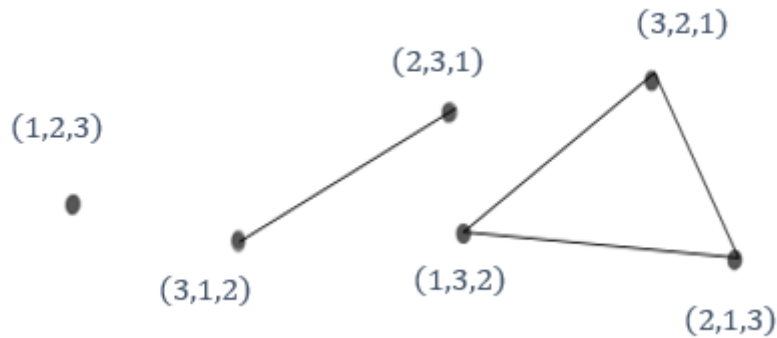


Figure 2

The order- graph $\Omega_{(S_3, o)}$ over the symmetric group (S_3, o) .

2. NEW RESULT

We will bring the following results:

Theorem 1:

For every group $(G, *)$, of order $n, n \in \mathbb{N}$, the number of the bondage element in the order- graph $\Omega_{(G, *)}$ over the group $(G, *)$ always will not exceed the number divisors of n . In addition, each bond element is a complete graph.

Theorem 2:

For all natural number $n, n > 1$,

- 1) The Order- graph $\Omega_{(Z_n, +_n)}$ over the group $(Z_n, +_n)$, always not connected.

In addition, if n is a prime, then:

- 2) The number of edges in the order- graph $\Omega_{(Z_n, +_n)}$ over the group $(Z_n, +_n)$ is $\frac{(n-1)(n-2)}{2}$.
- 3) The chromatic number for the order- graph $\Omega_{(Z_n, +_n)}$ over the group $(Z_n, +_n)$ always equals $n - 1$. i.e. $\chi(\Omega_{(Z_n, +_n)}) = n - 1$.

Moreover, if n is a prime, and, $n \geq 7$, then:

- 4) The order- graph $\Omega_{(Z_n, +_n)}$ over the group $(Z_n, +_n)$ is not planar.

Proof of Theorem 1:

Let $(G, *)$ be a group that his order is $n > 1, n \in \mathbb{N}$, and let $\Omega_{(G, *)}$ the order- graph over the group $(G, *)$. Denote by λ to the number of the divisors of n , mark them by the set,

$S = \{s_1, s_1, \dots, s_\lambda\}$ and denote by θ to the number of the bondage element in $\Omega_{(G,*)}$. We will prove that $\theta \leq \lambda$.

From the fact $0(\vartheta_i) / |V_{\Omega_{(G,*)}}|$, we got that for every $\vartheta_i \in V_{\Omega_{(G,*)}}, 1 \leq i \leq n$ that $o(\vartheta_i) \in S$. This means that there is a $j, 1 \leq j \leq \lambda$ in which $o(\vartheta_i) = s_j$. This means also that the set vertices of the graph $\Omega_{(G,*)}$ can be split into at most λ subsets $V_1, V_1, \dots, V_\lambda$ and each subset contains only vertices of the same order s_j .

In addition every $V_j, (1 \leq j \leq \lambda)$ is a bonding component, otherwise two vertices of different order can be connected on an edge, from here the number of the bondage element shall not exceed the number divisors of n . It means that $\theta \leq \lambda$.

Let us also note that every bonding component $V_j, (1 \leq j \leq \lambda)$, contains within it only vertices that are interconnected since they have the same order, and if we mark in $|V_j|$ to the vertex number in V_j then every bonding component in $\Omega_{(G,*)}$, none other than the complete graph $K_{|V_j|}$.

This completes the proof of theorem 1.

Example 2:

Let us now demonstrate the main argument in theorem 1, that the number of the bondage element (θ) in the order- over a group always will not exceed the divisors number (λ) of the order of the group (n) i.e. to see that $\theta \leq \lambda$.

To illustrate this, let us look at the order- graph $\Omega_{(Z_{14}, +_{14})}$ over the group $(Z_{14}, +_{14})$, (See figure 3). Here the number of divisors of the number 14 is four, $S = \{1,2,7,14\}$ i.e. ($\lambda = 4$), and number of the bondage element is four, ($\theta = 4$). Pay attention to each bond element is a complete graph, that they K_6, K_6, K_1 and K_1 .

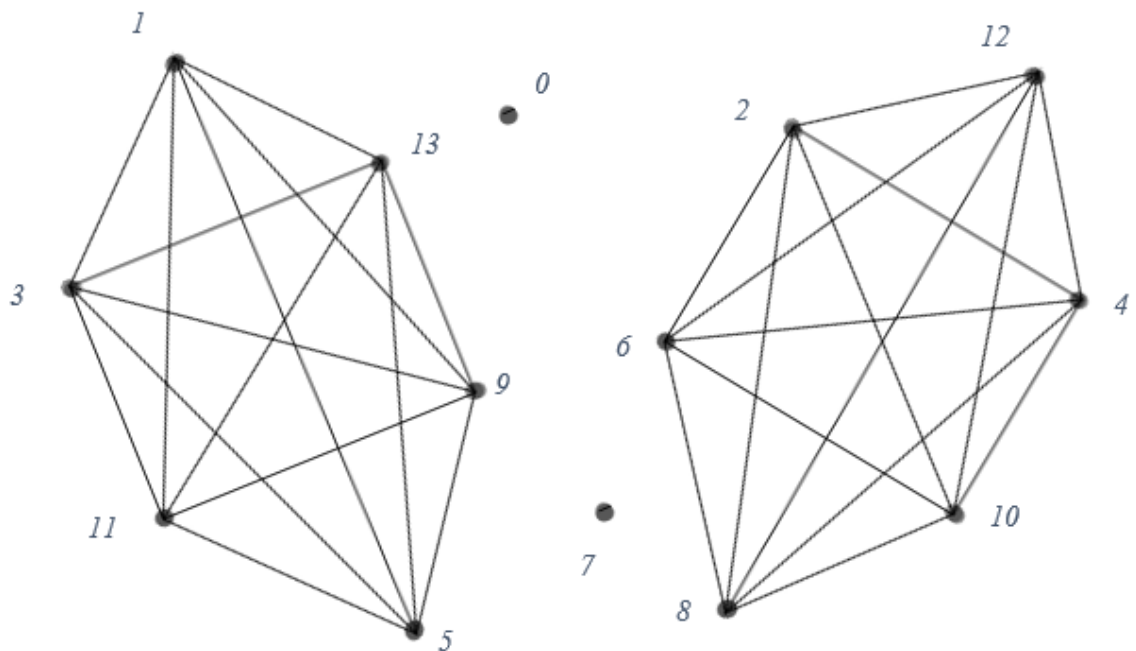


Figure 3.

Proof of Theorem 2:

Let $n > 1$ a natural number, and let $(Z_n, +_n)$ be the cyclic Group modulo n , and $\Omega_{(Z_n, +_n)}$ the order-graph over the group $(Z_n, +_n)$. To prove that graph $\Omega_{(Z_n, +_n)}$ is not connected, it is enough to find two vertices, which there is no path that connects them.

Let us look at both vertices 0 and $(n - 1)$, it is clear that $O(0) = 1$, (since that $0 = e_{(Z_n, +_n)}$), and $O(n - 1) = n$, (since that $(n, n - 1) = 1$).

Now the vertex 0 can only connect to vertices has order one, while the vertex $(n - 1)$ can only connect to vertices whose order is n , hence the two vertices 0 and $(n - 1)$, cannot connected.

This proves (1).

In addition, if n is a prime number then in this case $O(1) = O(1) = O(2) = \dots = O(1) = n$, because $(n, i) = 1$ for every $2 \leq i \leq n - 1$ when n is prime number, and it is clear that $O(0) = 1$.

Hence $\Omega_{(Z_n, +_n)}$ consists of two components of bonds one of them contains one vertex and the other contains $(n - 1)$ vertically connected vertices (i.e. the complete graph K_{n-1}), and the last component contains $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ edges that are all the edges of the graph.

This proves (2).

Note that, the complete graph K_{n-1} needs $(n - 1)$ colors to color its vertexes, because every two vertexes have an edge between them, so they do not get the same color i.e. $\chi(\Omega_{(Z_n, +_n)}) = n - 1$.

This proves (3).

Moreover, if n is a prime number, and $n \geq 7$ then K_6 one of the two bonding components of the order-graph $\Omega_{(Z_n, +_n)}$ over the group $(Z_n, +_n)$, and K_6 is not planar.

This proves (4).

This completes the proof of theorem 2.

Example 3:

Figure 4 expresses the order-graph $\Omega_{(Z_3, +_3)}$ over the group $(Z_3, +_3)$, and Figure 5 expresses the order-graph $\Omega_{(Z_5, +_5)}$ over the group $(Z_5, +_5)$, note that both are planar.



Figure 4

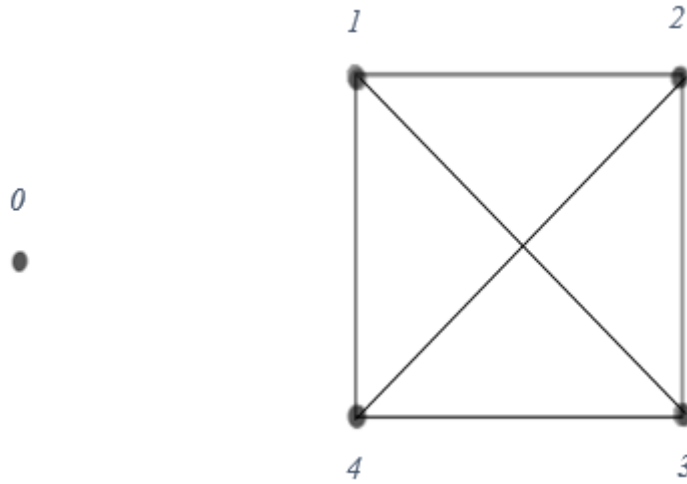


Figure 5

Corollary 1:

For all cyclic groups $(G,*)$, of order $n, n > 1$ the order- graph over it will not be connected. If n is a prime number, then the graph $\Omega_{(G,*)}$ contains $\frac{(n-1)(n-2)}{2}$ edges, and $\chi(\Omega_{(G,*)}) = n - 1$. In addition, if n is a prime number, $n \geq 7$, then the order- graph $\Omega_{(G,*)}$ is not planar.

Proof of Corollary 1:

This corollary is a direct result from the known theorem, which promises that every cyclic group $(G,*)$ of order n isomorphic to the group $(Z_n, +_n)$. It will lead to that every order- graph $\Omega_{(G,*)}$ is isomorphic to $\Omega_{(Z_n, +_n)}$. In addition, from the previous Theorem, (Theorem 1) we get correctness_Corollary 1.

Remark 3:

The following table describes the first cases for the order- graph $\Omega_{(Z_n, +_n)}$ over the group $(Z_n, +_n)$.

The order-graph over $(Z_n, +_n)$	Graph description	number of the bondage element (θ)	(λ) the number of the divisors of n	$\chi(\Omega_{(Z_n, +_n)})$	$ E_{(\Omega_{(Z_n, +_n)})} $
$(Z_2, +_n)$	$K_1 \cup K_1$	2	2	1	0
$(Z_3, +_n)$	$K_1 \cup K_2$	2	2	2	1
$(Z_4, +_n)$	$K_1 \cup K_2$	3	3	2	2
$(Z_5, +_n)$	$K_1 \cup K_4$	2	2	4	6
$(Z_6, +_n)$	$K_1 \cup K_1 \cup K_2 \cup K_2$	4	4	2	2
$(Z_7, +_n)$	$K_1 \cup K_6$	2	2	6	15
$(Z_8, +_n)$	$K_1 \cup K_1 \cup K_2 \cup K_4$	4	4	4	8
$(Z_9, +_n)$	$K_1 \cup K_2 \cup K_6$	3	3	6	16
$(Z_{10}, +_n)$	$K_1 \cup K_1 \cup K_4 \cup K_4$	4	4	4	12
$(Z_{11}, +_n)$	$K_1 \cup K_{10}$	2	2	10	45
$(Z_{12}, +_n)$	$K_1 \cup K_2 \cup K_2 \cup K_4$	4	5	4	10
$(Z_{13}, +_n)$	$K_1 \cup K_{12}$	2	2	12	66
$(Z_{14}, +_n)$	$K_1 \cup K_6 \cup K_6$	3	4	6	30
$(Z_{15}, +_n)$	$K_1 \cup K_1 \cup K_2 \cup K_2 \cup K_{11}$	5	3	11	57

3. CONCLUSIONS

After having the result mentioned above proven, we can refer to the following issues in question:

- What is the total number of edges in the order- graph $\Omega_{(Z_n, +_n)}$, and another order- graph?

- What is the chromatic number in the order- graph $\Omega_{(z_n, +n)}$, and another order- graph?

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