Chinese Journal of Geotechnical Engineering

Volume 42 Year 2020

Polynomial Coloring Applications

Wafiq Hibi
Assistant Professor
Head of the Mathematics Department in Sakhnin College.
The Academic College of Sakhnin

Keywords: Color polynomial, George Birkhoff formula, chromatic number, order of graph.

DOI: 10.11779/CJGE2020.42

ABSTRACT: Appropriate coloring to the vertices of a given graph G is to give color to each vertex so that two vertices connected by an edge will not have the same color. The smallest number of appropriate coloring called the chromatic number, and is often denoted by $\chi(G)$.
The number of ways to color the vertices of a given graph G in λ colors, given by placing λ in $\chi(G,\lambda)$, which is called the color polynomial of the graph G and which maintains:
$\chi(G,\lambda)=\chi(G-e,\lambda)-\chi(G/e,\lambda)$, Where $G-e$ is the graph without the edge e, and G/e is the graph obtained from the omitting of the edge e and shrinking the two vertices that apply to it into one vertex. (due [10]).
In this paper, we will expand on the relationship between the color polynomial and the order of the graph, while giving explicit formulas to the color polynomial.

1. INTRODUCTION

Definition 1: A graph $G:(V,E)$ is called k - coloring if we can color all his vertices with k colors so that no two vertices share the same edge. (For any coloring option, it is called proper coloring).

The smallest number of colors k , we call it the chromatic number, and often denoted by $\chi(G)$.

The number of ways to color the vertices of a given graph G in λ colors, denote $\chi(G,\lambda)$, and it is called the color polynomial of G.

George Birkhoff formula :(due [2]).

The color polynomial of the graph G given by placing λ the number of ways to color the vertices of G,$\chi(G,\lambda)=\chi(G-e,\lambda)-\chi(G/e,\lambda)$ where $G-e$ is the graph without the edge e, and G/e is the graph obtained from the omitting of the edge e and shrinking the two vertices that apply to it into one vertex.

Definition 2: The order of a graph is the number of vertices in the graph.

Definition 3: Monic polynomial is a polynomial whose leading coefficient is one. That is, it is of the form $x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$.

Remark 1 :(due [9]):
In this remark, we will mention some color polynomials graph families:
N_n , The blank graph, (A graph with n vertices without any edges). λ^n
T_n, (A graph with n vertices, which is connected, and without circles). $\lambda(\lambda-1)^{n-1}$.
C_n, , (A graph with n vertices terms on a simple circle). $(\lambda-1)^n+(-1)^n(\lambda-1)$.
K_n , the complete graph (the graph with n vertices where each vertex is connected to all the other vertexes).$\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-(n-1))$.
P_n, The path graph (a graph with n vertices without any two successive vertices are connected by an edge), $\lambda(\lambda-1)^{n-1}$.

Example 1:
In this example, we will find the color polynomial for:

(a) The tree graph.
If we choose a vertex ϑ in the tree, we can color it in any color that we have, from here we can do this in λ options, we will continue to color the neighboring vertices of ϑ we can do this in $\lambda - 1$ options, and so we will continue until all the vertices of the tree are colored, hence the coloring polynomial will be $\lambda(\lambda-1)^{n-1}$.

(b) The circle graph.
Based on George Birkhoff formula the color polynomial of graph G given by placing λ the number of ways to color the vertices of G on of $\chi(G,\lambda) = \chi(G-e,\lambda) - \chi(G/e,\lambda)$, so in the circle graph we have $\chi(C_n,\lambda) = \chi(C_n - e,\lambda) - \chi(C_n/e,\lambda)$.

It is clear that if we subtract from the graph of the circle C_n (a simple circle with n vertices and n edges) some edge we will get $C_n - e$ a tree T_n (a simple path with n vertices and $n-1$ edges).
It is clear so that C_n/e an $(n-1)$ order circle.

Now we consider $\chi(C_n,\lambda)$ recursively
$\chi(C_n,\lambda) = \chi(C_n - e,\lambda) - \chi(C_n/e,\lambda)$
$= \chi(T_n,\lambda) - \chi(C_{n-1}/e,\lambda)$
$= \chi(T_n,\lambda) - \chi(T_{n-1},\lambda) - \chi(C_{n-2}/e,\lambda)$
$=\chi(T_n,\lambda) - \chi(T_{n-1},\lambda) - \chi(T_{n-2},\lambda)\chi - (C_{n-3}/e,\lambda)$
$= \chi(T_n,\lambda) - \chi(T_{n-1},\lambda) - \chi(T_{n-2},\lambda) - \chi(C_{n-3}/e,\lambda)$
$= \chi(T_n,\lambda) - \chi(T_{n-1},\lambda) - \chi(T_{n-2},\lambda) - \cdots - \chi(T_{n-i},\lambda) - \cdots - \chi(T_4,\lambda) - \chi(C_3,\lambda)$
$= \sum_{i=4}^{n}(-1)^{n-i} \cdot \chi(T_i,\lambda) + (-1)^{n-1} \cdot \chi(C_3,\lambda)$

Now Considering $\sum_{i=4}^{n}(-1)^{n-i} \cdot \chi(T_i,\lambda)$:
Relying on (a) we get:

$$\sum_{i=4}^{n}(-1)^{n-i} \cdot \chi(T_n,\lambda)$$
$$= \sum_{i=4}^{n}(-1)^{n-i} \cdot \lambda \cdot (\lambda-1)^{i-1}$$
$$= \frac{\lambda}{\lambda-1} \cdot (-1)^n \sum_{i=4}^{n}(-1)^{-i} \cdot (\lambda-1)^i$$
$$= \frac{\lambda}{\lambda-1} \cdot (-1)^n \sum_{i=4}^{n}(1-\lambda)^i$$
$$= \frac{\lambda}{\lambda-1} \cdot (-1)^n \sum_{i=0}^{n-4}(1-\lambda)^{i+4}$$
$$= \frac{\lambda(1-\lambda)^4}{\lambda-1} \cdot (-1)^n \sum_{i=0}^{n-4}(1-\lambda)^i$$
$$= (-1)^{n-1} \cdot \lambda \cdot (1-\lambda)^3 \cdot \lambda \frac{1-(1-\lambda)^{n-3}}{\lambda}$$
$$= (-1)^{n-1} \cdot (1-\lambda)^3 + (-1)^n \cdot (1-\lambda)^n$$

Now Considering $(-1)^{n-1} \cdot \chi(C_3,\lambda)$:
The Color Polynomial for $\chi(C_3,\lambda)$ it is $\lambda(\lambda-1)(\lambda-2)$, ($\lambda$ colors to the first, $\lambda-1$ colors to the second and $\lambda-3$ colors to the third), so $(-1)^{n-1} \cdot \chi(C_3,\lambda) = (-1)^{n-1} \cdot \lambda(\lambda-1)(\lambda-2)$.
In summary, we will receive:
$\chi(C_n,\lambda) = \sum_{i=4}^{n}(-1)^{n-i} \cdot \chi(T_i,\lambda) + (-1)^{n-1} \cdot \chi(C_3,\lambda)$
$=(-1)^{n-1} \cdot (1-\lambda)^3 + (-1)^n \cdot (1-\lambda)^n + (-1)^{n-1} \cdot \lambda(\lambda-1)(\lambda-2)$.

New result:
As an extension, we will prove the following two lemmas:

Lemma 1:
The degree of the color polynomial in each graph equal to the order of the graph.

Lemma 2:
The color polynomial in each graph it is a Monic polynomial.

Proof of Lemma 1:
Let $G = (V, E)$ be a graph, denote that $|E| = m$ and $|V| = n$, the order of the graph.
We will prove that the degree of the color polynomial in each graph equal to the order of the graph, in induction on m the number of edges of the graph.
For $m = 0$, which means that the graph is empty without edges, and for each λ number of color, the color polynomial will be λ^n because each vertex can be colored in any of the colors.
Suppose the correctness of the lemma for any graph with a number of edges less than m , we will prove the correctness for a graph that has m edges.
From George Birkhoff formula the color polynomial of graph G given by placing λ the number of ways to color the vertices of G on $\chi(G,\lambda) = \chi(G-e,\lambda) - \chi(G/e,\lambda)$:
The Graph $G-e$ contains $m-1$ edges and n vertex, Therefore from the induction assumption the degree of color polynomial for its order is n, we will mark this polynomial
$\chi(G-e,\lambda) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in R, 0 \le i \le n$.
The Graph G/e contains $m-1$ edges and $n-1$ vertex, Therefore from the induction assumption the degree of color polynomial for its order is $n-1$, we will mark this polynomial
$\chi(G/e,\lambda) = b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$, $b_i \in R, 0 \le i \le n$.
So,
$\chi(G,\lambda) = a_n x^n + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_1 - b_1)x + (a_0 - b_0)$.
Then the degree of the color polynomial equal to the order of the graph.
This complete the proof of lemma 1

Proof of Lemma 2:
Let $G = (V, E)$ be a graph, denote that $|E| = m$ and$|V| = n$, the order of the graph.
Also here we will prove in induction on m the number of edges of the graph that the color polynomial in each graph it is a monic polynomial.
For $m = 0$, which means that the graph is empty without edges, and for each λ number of color the color polynomial will be λ^n (because each vertex can be color in any of the colors), then the color polynomial a monic.
Suppose the correctness of the lemma for any graph with a number of edges less than m , we will prove the correctness for a graph that has m edges.
From George Birkhoff formula the color polynomial of graph G given by placing λ the number of ways to color the vertices of G on $\chi(G,\lambda) = \chi(G-e,\lambda) - \chi(G/e,\lambda)$:
The Graph $G-e$ contains $m-1$ edges and n vertex, Therefore from the induction assumption the degree of color polynomial for its order is n, we will mark this polynomial
$\chi(G-e,\lambda) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in R, 0 \le i \le n$.
The Graph G/e contains $m-1$ edges and $n-1$ vertex, Therefore from the induction assumption the degree of color polynomial for its order is $n-1$, we will mark this polynomial
$\chi(G/e,\lambda) = x^{n-1} + \cdots + b_1 x + b_0$, $b_i \in R, 0 \le i \le n$.
So,
$\chi(G,\lambda) = x^n + (a_{n-1} - 1)x^{n-1} + \cdots + (a_1 - b_1)x + (a_0 - b_0)$.
The color polynomial in each graph it is a Monic polynomial.
This complete the proof of lemma 2

References

1. Alon, N. (2003), "A simple algorithm for edge-coloring bipartite multigraphs", *Information Processing Letters*, 85 (6): 301–302.
2. Birkhoff, George David (1912). A determinant formula for the number of ways of coloring a map. *Ann. Math*. 14 (1/4): 42–46.
3. Chen, G., Xingxing,. Y., Zang,. W. (2011), "Approximating the chromatic index of multigraphs", *Journal of Combinatorial Optimization*, 21 (2): 219–246.
4. Frieze, Alan M., Krivelevich, M., Sudakov, B. (2005), "The strong chromatic index of random graphs", *SIAM Journal on Discrete Mathematics*, **19** (3): 719–727.

5. *Giotis, I.; Kirousis, L.; Psaromiligkos, K. I.; Thilikos, D. M. (2015), "On the algorithmic Lovász Local Lemma and acyclic edge coloring", Proceedings of the Twelfth Workshop on Analytic Algorithmics and Combinatorics (ANALCO), p. 16,* doi:10.1137/1.9781611973761.2, ISBN 978-1-61197-376-1.
6. Hibi, W. (2021). Girth inequality in planar graphs. *The Journal of Multicultural Education.* (Accepted).
7. Hibi, W. (2017). Non-isomorphism between graph and its complement. *The Journal of Multicultural Education*, 7(6), 256-258.
8. Hibi, W. (2019). On regular graph, that has two types of faces that having two consecutive numbers of edges *Turkish Journal of Computer and Mathematics Education.* Vol.12 No.14 (2021), 1566– 1571.
9. Hibi, W. (2018). Relationships between faces in regular, connected and planar graphs. *The Journal of Multicultural Education.* (Accepted).
10. Hibi, W. (2017). The four-color theorem in the service of Euclidean distance into the $(n_0, \rho_0) - R^2$ graphs The Journal of Multicultural Education. (Accepted).
11. Makowsky, J. A.; Rotics, U.; Averbouch, I.; Godlin, B. (2006), "Computing graph polynomials on graphs of bounded clique-width", *Proc. 32nd Int. Worksh. Graph-Theoretic Concepts in Computer Science (WG 2006)*, Lecture Notes in Computer Science, 4271, Springer-Verlag, pp. 191–204.
12. Sokal, A. D. (2004), "Chromatic Roots are Dense in the Whole Complex Plane", *Combinatorics, Probability and Computing*, 13 (2): 221–261.